



# STABILITY OF LINEAR ALMOST-HAMILTONIAN PERIODIC SYSTEMS†

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A linear Hamiltonian system with periodic coefficients is subject to a small "dissipative" perturbation that makes it asymptotically stable. The conditions under which the perturbation remains dissipative for all Hamiltonian systems sufficiently close to the original one are discussed. © 1997 Elsevier Science Ltd. All rights reserved.

## 1. STATEMENT OF THE PROBLEM AND MAIN RESULTS

Suppose that a linear Hamiltonian system of differential equations  $\langle H_0 \rangle$  with periodic coefficients is stable—each of its solutions is bounded. Any stable system is strongly stable if none of its multipliers has multiplicity greater than one.‡ If there are multiple multipliers, further conditions are necessary to ensure strong stability. Such conditions, both necessary and sufficient, have been found by Krein, Gel'fand and Lidskii (see [1, Ch. 3], [2, Sec. 42] and Section 2C below).

Let us assume now that the stable system  $\langle H_0 \rangle$  is perturbed by small ( $\sim \varepsilon$ ) terms that leave it linear and periodic. We will say that such a perturbation is dissipative for  $\langle H_0 \rangle$  if it makes the system asymptotically stable. We will call the perturbation strongly dissipative if the above property is maintained over the whole neighbourhood of the original Hamiltonian system. If all multipliers of the system  $\langle H_0 \rangle$  are distinct, dissipativeness (with a slightly sharper definition) implies strong dissipativeness. When there are multiple multipliers, additional conditions are needed. A sufficient condition was established in [3], where both Hamiltonian and non-Hamiltonian perturbations of the same order as  $\varepsilon$  are considered to be small (see Section 7 below).

The main purpose of this paper is to show that this condition is also necessary. To be precise: it is almost necessary and sufficient, to the extent that this is usually possible for stability conditions: if strict inequalities (of the  $<$  type) are sufficient, then non-strict inequalities ( $\leq$ ) are necessary for stability. This statement holds for both versions of the definition—that used in [3] and that presented below. We will now give a rigorous formulation.

Let us write the initial Hamiltonian system  $\langle H_0 \rangle$  as

$$\frac{dx}{dt} = JH_0(t)x; \quad H_0(t) \equiv H_0(t+T), \quad J = \begin{vmatrix} 0 & I_n \\ -I_n & 0 \end{vmatrix} \quad (1.1)$$

where (for each  $t$ )  $H_0(t)$  is a symmetric matrix,  $I_n$  is the  $n \times n$  identity matrix and  $x \in \mathbb{R}^{2n}$ .

Let  $\Phi_0(t)$  be a fundamental matrix of solutions of system (1.1) such that  $\Phi_0(0) = I_{2n}$ . Let  $\Phi_0 = \Phi_0(T)$  denote the operator of the mapping in a period (monodromy operator) and  $\mu_k$  its eigenvalues—the multipliers of system (1.1). System (1.1) is stable if  $|\mu_k| = 1$  for all  $k$  and  $\Phi_0$  has a complete set of eigenvectors—the matrix  $\Phi_0$  can be reduced to diagonal form. (As we shall be using the standard bases in  $\mathbb{R}^m$  and  $\mathbb{C}^m$ , we will not use different notation for linear operators and their matrices.)

Define a  $\delta$ -neighbourhood of system (1.1) to be the set of all Hamiltonian systems  $x = JH(t)x$  such that  $\|H(t) - H_0(t)\| < \delta$  ( $0 \leq t \leq T$ ),  $H(t) \equiv H(t+T)$ , where  $\|\cdot\|$  is some matrix norm. (Instead of uniform closeness of  $H$  and  $H_0$  one might require closeness in the mean: smallness of the integral if  $\|H(t) - H_0(t)\|$  over a period.)

System (1.1) is said to be strongly stable if, for some  $\delta > 0$ , all the systems in its  $\delta$ -neighbourhood are stable.

Let us consider a "perturbed" system with the same period

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‡The definitions will be recalled below.

$$\dot{x} = JH_0(t)x + D(t, \epsilon)x; \quad D(t, \epsilon) = \epsilon D(t) + o(\epsilon) \tag{1.2}$$

*Definition 1.* The perturbation  $D(t, \epsilon)x$  is said to be dissipative for system (1.1) if system (1.2) is asymptotically stable for all  $\epsilon$  in  $0 < \epsilon < \epsilon_*$  (all its multipliers lie inside the unit circle).

If system (1.1) admits of a dissipative perturbation, then  $|\mu_k| = 1$  for all  $k$ , but the system need not be stable (the solutions may increase only weakly, linearly in  $t$ ). "Normally", in a dissipative perturbation the distance  $\mu_k(\epsilon)$  from the circle  $|\mu| = 1$  is of the order of  $\epsilon$ . In order to exclude degenerate cases, where the answer may depend on (unwritten) terms  $o(\epsilon)$ , we will incorporate this requirement in the definition.

*Definition 1a.* A perturbation  $D(t, \epsilon)x$  is dissipative in the first approximation if all the multipliers of system (1.2) satisfy the inequality  $|\mu_k(\epsilon)| < 1 - c\epsilon$  (where  $c < 0$  is independent of  $\epsilon$ ).

*Definition 2.* A perturbation  $D(t, \epsilon)x$  is strongly dissipative for system (1.1) if it is dissipative for any system in some  $\delta$ -neighbourhood of (1.1).

*Remark 1.* For system (1.1) to admit of a strongly dissipative perturbation it must be strongly stable. Indeed, if there is no strong (Hamiltonian) stability, then any  $\delta$ -neighbourhood of (1.1) contains an exponentially unstable system (that is, a system with multipliers outside the unit circle). Instability is preserved for sufficiently small  $\epsilon$ .

In order to establish conditions for dissipativeness, we will examine how the eigenvalues of the monodromy operator  $\Phi$  are affected by a small perturbation. To simplify matters, we shall assume that  $D(t, \epsilon)$  is an analytic function of  $\epsilon$ . We shall also assume, without special mention, that the matrix functions  $H(t)$  and  $D(t, \epsilon)$  are periodic functions of  $t$  with period  $T$  (and also continuous or piecewise continuous).

## 2. AUXILIARY INFORMATION

### A. Transition to the complex domain (see [1])

To change from  $\mathbb{R}^m$  to  $\mathbb{C}^m$  ( $m = 2n$ ), we complete the definition of the linear operators  $A$  in the usual way:  $Az = Ax + iAy$  for  $z = x + iy$  ( $x, y \in \mathbb{R}^m$ ). When that is done,  $A\bar{z} = \bar{A}z$ , ( $\bar{z} = \bar{x} - iy$ ). Let  $(x, y)$  be the standard scalar product in  $\mathbb{R}^m$ . In  $\mathbb{C}^m$ , define

$$(z, w) = \sum_{k=1}^m z^k \overline{w^k}; \quad z = (z^1, \dots, z^m); \quad w = (w^1, \dots, w^m) \tag{2.1}$$

The antisymmetric bilinear form  $[x, y] = (Jx, y)$ , which is invariant to the interchange of  $x$  and  $y$  since the system is Hamiltonian, becomes  $[z, w] = (Jz, w)$ ; the values of  $[z, z]$  are then pure imaginary. We define

$$\langle z, w \rangle = i[z, w] = i(Jz, w) \tag{2.2}$$

The quadratic (Hermitian) form  $v(z) = \langle z, z \rangle$  is real and non-degenerate, but of variable sign. Let  $A^+$  denote the operator adjoint to  $A$  in the sense of (2.2):  $\langle Az, w \rangle = \langle z, A^+w \rangle$ . If  $A$  is a real matrix, then  $A^+ = -JA^*J$ , where  $A^*$  is the transpose.

The monodromy operator  $\Phi$  for a Hamiltonian system preserves  $[x, y]$  (the matrix of  $\Phi$  is symplectic). In  $\mathbb{C}^m$  we obtain  $\langle \Phi z, \Phi z \rangle = \langle z, w \rangle$ , that is,  $\Phi$  is a unitary operator in the sense of (2.2),  $\Phi^+ = \Phi^{-1}$ . If  $e$  is an eigenvector,  $\Phi e = \mu e$ , then  $\langle \Phi e, \Phi e \rangle = |\mu|^2 \langle e, e \rangle$ . Hence it follows that either  $|\mu| = 1$  or  $v(e) = 0$ . If  $|\mu| = 1$  but  $\mu$  is a multiple eigenvalue associated with a Jordan cell, then again  $v(e) = 0$ . Indeed, in that case there is a vector  $f$  such that  $\Phi f - \mu f = e$ . Then  $\langle e, e \rangle = (\bar{\mu}^{-1} - \mu) \langle f, e \rangle = 0$ . If  $|\mu| = 1$  and  $\mu$  is a simple eigenvalue, then  $v(e) \neq 0$ . Let  $\Phi e_k = \mu_k e_k$ ; then  $\langle e_1, e_2 \rangle = 0$  if  $\mu_1 \mu_2^{-2} \neq 1$ .

### B. Lemmas from perturbation theory for linear operators in $\mathbb{C}^m$ [4]

*Lemma 1.* Let  $E_0$  be an  $s$ -dimensional invariant subspace of an operator  $A_0$ , corresponding to an eigenvalue  $\lambda$  of multiplicity  $s$ . Let  $A(\epsilon) = A_0 + \epsilon B$ . Then for  $|\epsilon| < \epsilon_*$  the operator  $A(\epsilon)$  has an invariant subspace  $E_\epsilon$  close to  $E_0$ : if  $P_\epsilon$  is the operator of projection onto  $E_\epsilon$  ( $P_\epsilon A(\epsilon) = A(\epsilon) P_\epsilon$ ), then  $\|P_\epsilon - P_0\| \leq C\epsilon$ .

If all the eigenvalues of  $A_0$  are different, then  $\lambda_k(\epsilon)$  are analytic functions of  $\epsilon$  in the neighbourhood of  $\epsilon = 0$ . When there are multiple eigenvalues  $\lambda_k(0)$ , that is not the case. But if  $A_0$  has no Jordan cells then, to the first order in  $\epsilon$ , the formulae of perturbation look the same as they would have been in the absence of multiplicity.

*Lemma 2.* Let  $A_0$  have an  $s$ -fold eigenvalue  $\lambda$ , associated with an  $s$ -dimensional invariant subspace  $E_0$  of eigenvectors. Then for  $|\epsilon| \ll 1$ ,  $A(\epsilon)$  has eigenvalues  $\lambda_k(\epsilon)$  such that

$$\lambda_k(\epsilon) = \lambda + \epsilon\gamma_k + o(\epsilon), \quad k = 1, \dots, s \tag{2.3}$$

where  $\gamma_k$  are the eigenvalues of the operator  $B_1 = PB$  on  $E_0$  (the latter is an invariant subspace of this operator too and  $P$  is a projector onto  $E_0$ ). Each  $\gamma_k$  in formulae (2.3) is repeated a number of times equal to its multiplicity.

*Refinements.* 1. Some of the  $\lambda_k(\epsilon)$  may be identically multiple:  $\lambda_k(\epsilon) \equiv \lambda_j(\epsilon)$ —the perturbation does not necessarily remove the degeneracy entirely.

2. The lemmas remain valid when  $B$  is replaced by an analytic function  $B(\epsilon)$  of  $\epsilon$ ; in that case  $B_1 = P_0B(0)$ .

*C. Conditions of strong stability for Hamiltonian systems [1]*

Let  $\Phi_0$  be the monodromy operator of system (1.1) and let  $\mu_1, \dots, \mu_l$  be the set of (pairwise distinct) multipliers ( $1 \leq 2n$ ). Let  $E_j$  be the invariant subspace associated with  $\mu_j$ .

*Theorem (Krein–Gel’fand–Lidskii).* A necessary and sufficient condition for system (1.1) to be strongly stable is that the form  $v(z) = \langle z, z \rangle$  be definite on each of the subspaces  $E_j$ .

*Remark 2.* It follows from the conditions of the theorem (see  $A$ ) that  $|\mu_j| = 1$  for each  $j$  and  $\Phi_0$  has no Jordan cells. If the conditions of the theorem hold for  $\Phi_0$ , it is also true for symmetric matrices close to  $\Phi_0$ . Hence it is clear that the conditions are indeed sufficient. Note that the conditions of the theorem exclude real  $\mu_j$ .

3. DISSIPATIVE PERTURBATIONS IN THE CASE OF SIMPLE MULTIPLIERS

Suppose that all the multipliers of system (1.1) are simple and the system is stable:  $\Phi_0 e_k = \mu_k e_k$ ;  $k = 1, \dots, 2n$ ;  $|\mu_k| = 1$ . The multipliers of system (1.2) may be written as

$$\mu_k(\epsilon) = \mu_k + \epsilon\gamma_k + O(\epsilon^2)$$

Put  $r_k = \text{Re}(\mu_k\gamma_k)$ . If  $r_j < 0$ , then (for small  $\epsilon$ )  $|\mu_j(\epsilon)| < 1$ . A sufficient condition for  $d(t, \epsilon)x$  to be dissipative is that, for all  $j$ , we have  $r_j < 0$ ; a necessary condition is that each of the inequalities  $r_j \leq 0$  holds. If the inequalities  $r_j < 0$  hold for  $H_0(t)$ , they remain true for nearby functions  $H(t)$ .

Thus: if a perturbation  $D(t, \epsilon)x$  for system (1.1) is dissipative in the first approximation, it is strongly dissipative (see the definitions in Section 1).

Let us write down the formulae more explicitly. Let the monodromy operator of system (1.2) have the form  $\Phi_\epsilon = \Phi_0 + \epsilon\Psi_0 + o(\epsilon)$ . We have the usual expressions for  $\gamma_k$ :  $\gamma_k = \langle \Psi_0 e_k, e_k \rangle / \langle e_k, e_k \rangle$  (we recall that here  $\langle e_k, e_k \rangle \neq 0$  for all  $k$ ). We note that  $\bar{\mu}_k \langle \Psi_0 e_k, e_k \rangle = \langle \Phi_0^{-1} \Psi_0 e_k, e_k \rangle$  and rewrite the conditions  $r_k < 0$  ( $k = 1, \dots, 2n$ ) as

$$Q_v(e_k) < 0; \quad Q_v(z) = \langle z, z \rangle \text{Re} \langle \Phi_0^{-1} \Psi_0 z, z \rangle \tag{3.1}$$

Assuming that the fundamental matrix  $\Phi_0(t)$  is known, we have a formula for  $V_0 = \Phi_0^{-1} \Psi_0$

$$\Phi_\epsilon = \Phi_0 + \epsilon\Psi_0 + o(\epsilon); \quad V_0 = \int_0^T \Phi_0^{-1}(t) D(t) \Phi_0(t) dt \tag{3.2}$$

Thus, in the case of simple multipliers everything is indeed simple. We now proceed to the main problem—investigating the case of multiple multipliers.

*Remark 3.* If there is a simple multiplier  $\mu_j$  (nothing being known about the others), then the condition  $Qv(e_j) \leq 0$  is obviously necessary for  $D(t, \epsilon)x$  to be dissipative.

4. SUFFICIENT CONDITIONS FOR STRONG DISSIPATIVENESS

Let system (1.1) be strongly stable (see Sections 1 and 2C). Then each multiplier  $\mu_k(\epsilon)$  of system (1.2) is “generated” by some multiplier  $\mu_j$  of system (1.1) (see Lemma 2)

$$\mu_k(\epsilon) = \mu_j + \epsilon\gamma_k + o(\epsilon), \quad j = j(k)$$

Fixing  $j$ , consider the group of multipliers  $\mu_k(\epsilon)$  generated by a single  $\mu = \mu_j$ . Let  $E = E_j$  be the corresponding invariant subspace of  $\Phi_0$  and  $P = P_j$  the projector onto  $E_j$ ,  $P\Phi_0 = \Phi_0P$ . The numbers  $\gamma_k$  are the eigenvalues of  $P\Psi_0$  on  $E$  (for  $\Psi_0$  see (3.2)). All  $\mu_k(\epsilon)$  in this group lie within the unit circle if  $\text{Re}(\mu\gamma_k) < 0$  for all  $k$ , i.e. all the eigenvalues of the operator  $K = \mu P\Psi_0$  (considered on  $E$ ) lie in the left half-plane. A sufficient condition for that to be true is  $\text{Re}\langle Kz, z \rangle < 0$  if  $v(z) = \langle z, z \rangle > 0$  on  $E$  (for  $z \neq 0$ ); if  $v(z) < 0$  on  $E$ , the inequality is reversed. We now observe that, as  $\Phi_0$  is a "unitary" operator, the projectors  $P$  are "self-adjoint":  $\langle Pz, w \rangle = \langle z, Pw \rangle$ . Therefore, for  $z \in E$ ,  $V_0 = \Phi_0^{-1}\Psi_0$

$$\langle Kz, z \rangle = \bar{\mu}\langle \Psi_0z, z \rangle = \langle \Psi_0z, \Phi_0z \rangle = \langle V_0z, z \rangle \tag{4.1}$$

Running through all  $E_j$ , we obtain:  $|\mu_k(\epsilon)| < 1$  for all  $k$ ,  $0 < \epsilon < \epsilon_0$ , of for any  $j$  we have

$$v(z) \neq 0 \text{ for all } z \in E_j (z \neq 0); v(z) = \langle z, z \rangle \tag{4.2}$$

$$Q_v(z) \equiv v(z)Q(z) < 0 \text{ for the same } z; Q(z) = \text{Re}\langle V_0z, z \rangle \tag{4.3}$$

Conditions (4.2) and (4.3) are also sufficient for strong dissipativeness. If we replace  $H_0(t)$  by a nearby function  $H(t)$  ( $\|H(t) - H_0(t)\| < \delta$ ), then  $\Phi_0$ ,  $\Psi_0$  and  $V_0$  are changed by an amount  $\leq C\delta$  ( $\Phi_0 \mapsto \Phi$ ,  $V_0 \mapsto V$ ). The subspaces  $E_j$  are replaced by nearby invariant subspaces  $E'_j$  of  $\Phi$ . (see Lemma 1). For sufficiently small  $\delta$ , inequalities (4.2) and (4.3) remain valid for  $V$  and  $E_j$ .

*Remark 4.* Every eigenspace  $E_k(\Phi)$  of  $\Phi$  is contained in some  $E'_j$ . Inequalities (4.2) and (4.3) ensure that  $D(t, \epsilon)x$  will be dissipative in relation to  $H(t)$  "with room to spare": they need only hold for each  $E_k(\Phi)$  separately.

*Remark 5.* Sufficient conditions for stability may be obtained more easily by constructing a "Lyapunov function"  $L$  for the mapping  $\Phi_\epsilon$  ( $L(\Phi_\epsilon z) < L(z)$ ) (see [3]). Our proof anticipates the analysis of the necessary conditions.

### 5. NECESSARY CONDITIONS FOR STRONG DISSIPATIVENESS

Conditions (4.2) and (4.3) are not necessary for  $D(t, \epsilon)x$  to be dissipative for a given  $H_0(t)$ . Indeed, the truth of the inequalities  $\text{Re}(\mu_j\gamma_k) < 0$  for all  $k$  and  $j = j(k)$  is equivalent to the requirement that  $Q_v(f) < 0$  for all eigenvectors  $f$  of the operators  $P_j\Phi_0$  (in the appropriate subspaces  $E_j$ ) and for them only. But if the dissipative property is invariant to arbitrarily small changes in  $H_0(t)$ , inequalities (4.2) and (4.3) (with a  $\leq$  sign in (4.3)) become necessary: any vector  $e$  in  $E_j$  may become an eigenvector of  $\Phi$  with a simple eigenvalue.

Thus, let the perturbation  $D(t, \epsilon)x$  be strongly dissipative for a given system (1.1) (see Definition 2), and fix some  $\delta$ . System (1.1) is strongly stable in the Hamiltonian sense (see Remark 1). Let us assume that in some subspace  $E_j$  of dimension greater than one (see Remark 3) there is a vector  $e$  such that  $Q_v(e) > 0$ . Define an operator  $\Phi$  as follows. Put  $\Phi e = \mu_\alpha e$  and  $\Phi \bar{e} = \bar{\mu}_\alpha \bar{e}$ , where  $\mu_\alpha = \mu_j \exp(i\alpha)$ ,  $\alpha > 0$ . Let  $E$  be the two-dimensional subspace spanned by  $e$  and  $\bar{e}$ . The operator  $\Phi$  is unitary on  $E$  in the sense of the "scalar product"  $\langle z, w \rangle$ , since  $e$  and  $\bar{e}$  are orthogonal (see the end of Section 2A). Consider an  $(m - 2)$ -dimensional subspace orthogonal to  $E$  and define there  $\Phi z = \Phi_0 z$ . Complete the definition of  $\Phi$  to  $\mathbb{C}^m$  by linearity;  $\Phi$  is unitary in  $\mathbb{C}^m$ ,  $\Phi \bar{z} = \overline{\Phi z}$  for all  $z$ . Therefore  $\Phi$  is a real symplectic matrix. The set of eigenvalues of  $\Phi$  consists of  $\mu_\alpha$ ,  $\bar{\mu}_\alpha$  and all  $\mu_j, j = 1, \dots, l$ .

*Lemma 3.* Let  $\Phi_0$  be generated by  $H_0(t)$ :  $\Phi_0$  is the monodromy matrix of the periodic Hamiltonian system  $x' = JH_0(t)x$ . Any symplectic matrix  $\Phi$  such that  $\|\Phi - \Phi_0\| < \eta$  is generated by a matrix function  $H(t)$  such that

$$\|H(t) - H_0(t)\| < C\eta$$

We omit the proof.

Now suppose that the operator  $\Phi$  defined above is generated by a symmetric matrix function  $H(t)$  and  $\|H(t) - H_0(t)\| \leq C\|\Phi - \Phi_0\| \leq C_2\eta$ . Choose  $\alpha$  so small that  $C_2\alpha$  is less than  $\delta$  and the eigenvalue  $\mu_\alpha$  is not multiple (is different from any  $\mu_j$ ). Now  $\|H(t) - H_0(t)\| < \delta$  and so the system  $x' = JH(t)x + D(t, \epsilon)x$  must be asymptotically stable for small  $\epsilon > 0$ . But this is impossible, since  $e$  is an eigenvector of  $\Phi$  belonging to a simple eigenvalue, and so  $Q_v(e) > 0$  (see Remark 3).

Thus, the condition  $Q_v(z) \leq 0$  for all vectors  $z$  of (any) subspace  $E_j$  is necessary for the system to be strongly dissipative. Instead of "all vectors of  $E_j$ ", one can also speak of all eigenvectors of  $\Phi_0$  (see Remarks 1 and 2).

6. FORMULATION OF THE THEOREM

Let  $x' = JH_0(t)x$  be a periodic Hamiltonian system,  $\Phi_0$  its monodromy operator and  $E_j$  the invariant subspaces for  $\Phi_0$  in  $C^m$  associated with the (pairwise distinct) eigenvalues  $\mu_j$ . Combining the material of Sections 4 and 5, we obtain the following.

*Theorem.* A perturbation  $D(t, \epsilon)x$  of system (1.1) is strongly dissipative in the sense of Definition 2 if, in any subspace  $E_j$ , the quadratic forms  $v(z)$  and  $Q(z)$  are sign-definite and have opposite signs:  $v(z)Q(z) < 0$  for  $z \in E_j, z \neq 0$ , where  $v(z) = \langle z, z \rangle$ ;  $Q(z) = \text{Re} \langle V_0 z, z \rangle$  and  $V_0$  is given by formula (3.2). The truth of the inequalities  $v(z) \neq 0, v(z)Q(z) \leq 0$  (for the same  $z$ ) is a necessary condition for strong dissipativeness.

The condition  $v(z) \neq 0$  relates to the unperturbed system (1.1). It implies that the system is strongly stable in the class of Hamiltonian systems (see Section 2). The form of the function  $Q(z)$  depends on the principal term of the perturbation  $\epsilon D(t)x$ . The condition  $Q(z) \neq 0$  means that the effect of the perturbation (for all  $H$  close to  $H_0$ ) is uniquely defined by  $D(t)$ . A more explicit expression for  $Q(z)$  is (see Section 2A and formula (3.2))

$$Q(z) = -\text{Im} \langle JV_0 z, z \rangle; \quad JV_0 = \int_0^T \Phi_0^*(t) JD(t) \Phi_0(t) dt \tag{6.1}$$

where we have used the equality

$$\Phi_0^{-1}(t) = \Phi_0^+(t) = -J\Phi_0^*(t)J.$$

7. FINAL REMARKS: DISCUSSION OF THE DEFINITIONS

Strong Hamiltonian stability may be defined by explicitly introducing small perturbations. We will say a system  $x' = JH_0(t)x$  is strongly stable if, for any (symmetric) matrix function  $H_1(t)$ ,  $\epsilon_*$  exists such that, whenever  $|\epsilon| < \epsilon_*$ , the system  $x' = J(H_0(t) + \epsilon H_1(t))x$  is stable. This definition is equivalent to the previous one. Once this definition is adopted, small non-Hamiltonian perturbations may be treated "on a par" with Hamiltonian perturbations.

*Definition 3* [3]. The system  $x' = JH_0(t)x$  (1.1) is said to be strongly stable under a given non-Hamiltonian perturbation  $d(t, \epsilon)x$  if, for any matrix function  $H_1(t)$  ( $H_1^* = H_1$ ),  $\epsilon_*$  exists such that the perturbed system (7.1) is stable for  $0 \leq \epsilon \leq \epsilon_*$ .

$$x' = J(H_0(t) + \epsilon H_1(t))x + D(t, \epsilon)x$$

$$D(t, \epsilon) = \epsilon D(t) + o(\epsilon) \tag{7.1}$$

The Hamiltonian perturbation in (7.1) may exceed the non-Hamiltonian perturbation as strongly as desired. Using this fact, one can prove that systems which are stable in the sense of Definition 3 are strongly stable in the Hamiltonian sense.

Definition 2 also yields a concept of stability under a perturbation.

*Definition 2a.* System (1.1) is strongly stable under a given perturbation  $D(t, \epsilon)x$  if (a) it is stable in the absence of perturbations (when  $\epsilon = 0$ ); (b) the perturbed system (1.2) is asymptotically stable for all  $\epsilon$  in some interval  $0 < \epsilon < \epsilon_*$ ; and (c) both properties are maintained in a certain  $\delta$ -neighbourhood of system (1.1).

In other words, a Hamiltonian system (1.1) is strongly stable under a given (non-Hamiltonian) perturbation  $D(t, \epsilon)x$  if the perturbation is strongly dissipative for the system.

Definition 3 is, at first glance, quite different from Definition 2a. In actual fact they are "almost equivalent". Indeed a system is stable in the sense of Definition 3 if inequalities (4.2) and (4.3) are true;

this sufficient condition was established previously [3] (in real notation, see Appendix). On the other hand, the non-strict inequalities (4.3) are necessary conditions for stability in the sense of Definition 3 (see Appendix). Thus, the same sufficient conditions and (very similar) necessary conditions "work" for both definitions. If one requires in addition that the answer—stable or unstable—be determined by  $D(t)$ , the strict inequalities (4.2) and (4.3) become necessary and sufficient in both cases.

A last comment—on uniform stability. The fact that all solutions of a linear homogeneous system of differential equations are bounded is equivalent to stability in Lyapunov's sense of its trivial (and any other) solution. For families of equations it is useful to go back to the original definition: the trivial solution of a system  $\dot{x} = A(t)x$  is stable if  $\|x(t)\| \leq C \|x(0)\|$ , where  $\|\cdot\|$  denotes some norm in the space of vectors  $x$ . The larger (the least possible) constant  $C$ , the "worse" the stability of the system.

A family of equations is uniformly stable if one can choose the same constant  $C$  for all equations of the family. It is natural to incorporate some such uniformity in the notion of strong stability: if the constant  $C$  is arbitrarily large for systems close to the given system, one could scarcely call the system "strongly stable". However, there is no need to include this requirement in the definition.

The following statements can be verified.

1. Let system (1.1) be strongly stable in the class of Hamiltonian systems. Then the stability is uniform in some neighbourhood of the system.

2. Let system (1.1) be stable in the sense of Definition 3; more precisely, suppose that conditions (4.2) and (4.3) are satisfied. Then, if  $\|H_1(t)\| \leq M$ ,  $\epsilon_* > 0$  exists such that, for  $0 \leq \epsilon \leq \epsilon_*$ , all the systems (7.1) are uniformly stable. A similar statement holds for Definition 2a.

### APPENDIX

#### A. Real notation

Let  $E$  be a subspace of  $\mathbb{C}^m$ . We define  $\text{Re } E \subset \mathbb{R}^m$  as follows: if  $z \in E$  and  $z = x + iy$ , then  $x, y \in \text{Re } E$ . If  $E$  is the eigenspace of the monodromy operator associated with an eigenvalue  $\mu = \alpha + i\beta$  ( $\beta \neq 0$ ), then  $\text{Re } E$  is associated with the pair  $\mu, \bar{\mu}$ .

The condition  $\langle z, z \rangle \neq 0$  on  $E$  is equivalent to the condition  $[\Phi x, x] \neq 0$  on  $\text{Re } E$ : if  $\Phi z = \mu z$ , then  $\langle z, z \rangle = 2\beta^{-1}[\Phi x, x]$ . For an arbitrary operator  $A$  we have  $\text{Re } \langle Az, z \rangle = \langle A_s z, z \rangle$ , where  $2A_s = A + A^+$ . If the operator  $A$  is defined by a real matrix, then, whenever  $\Phi z = \mu z$

$$\langle A_s z, z \rangle = 2\beta^{-1}[A_s \Phi x, x].$$

Conditions (4.2) and (4.3) may be rewritten as follows. For each  $j$ , it must be true that  $[\Phi x, x] \neq 0$  (4.2'),  $[V_s \Phi x, x] < 0$  (4.3') for all  $x \in \text{Re } E_j$  ( $x \neq 0$ ). The explicit form of condition (4.3') can be derived by using formula (3.2) for  $V$  (and the fact that  $A^+ = -JA^*J$ )

$$2[V_s \Phi x, x] = (J_1 \Phi x, x) \\ J_1 = \int_0^T \Phi^*(t)(JD(t) + D^*(t)J)\Phi(t)dt$$

This expression is identical (apart from notation) with the formula established in [3].

#### B. On conditions for stability in the sense of Definition 3

For the monodromy operator of system (7.1) we have (see (3.2))

$$\Phi_\epsilon = \Phi_0 + \epsilon \Psi_0 + o(\epsilon); \quad \Psi_0 = \Phi_0 V; \quad V = V_0 + V_1 \\ V_0 = \int_0^T \Phi_0^{-1}(t)D(t)\Phi_0(t)dt; \quad V_1 = \int_0^T \Phi_0^{-1}(t)JH_1(t)\Phi_0(t)dt$$

For any symmetric matrix function,  $H_1(t) V_1^+ = -V_1$ . Therefore,  $\text{Re } \langle Vz, z \rangle = \text{Re } \langle V_0 z, z \rangle$ . The rest of the proof that conditions (4.2) and (4.3) are sufficient for system (7.1) to be stable for  $0 < \epsilon \ll 1$  is identical with that presented in Sections 4 and 5 (see also Remark 5).

The proof of necessity may be sketched as follows: Suppose we know that system (7.1) is strongly stable under a given non-Hamiltonian perturbation. Let us assume that strong stability in the Hamiltonian sense has already been proved. Let  $\mu_1, \dots, \mu_l$  ( $l \leq 2n$ ) be the eigenvalues of  $\Phi_0$ . Once again (see Section 4), for each  $j \leq l$  we have a group of multipliers of system (7.1) for which  $\mu_k(\epsilon) = \mu_j + \epsilon \gamma_k + o(\epsilon)$ , where  $\gamma_k$  are the eigenvalues of the operator  $P\Psi_0$  in the subspace  $E_j$ , which depend on the choice of  $H_1, P = P_j$ . If system (7.1) is stable in the sense of Definition 3, then necessarily for any  $H_1(t)$ . In other words, all the eigenvalues of the operator  $P_j V$ , considered on  $E_j$ , which depend on the choice of  $H_1, P = P_j$ . If system (7.1) is stable in the sense of Definition 3, then necessarily  $\text{Re}(\bar{\mu} \gamma_k)$

$\leq 0$  for any  $H_1(t)$ . In other words, all the eigenvalues of the operator  $P_j V$ , considered on  $E_j$ , lie in the (closed) left half-plane for any  $H_1$ . We now note that  $V_1^+ = -V_1$  and that any such "anti-symmetric" operator is obtained for some  $H_1(t)$  (this is an "infinitesimal" analogue of Lemma 3). Fix  $j$  and prescribe  $E_j$  as an invariant subspace of  $V_1$ , with the following definition: on  $E_j$ , put  $V_1 = A, A^+ = -A$ , on the other  $E_j$ 's, put  $V_1 = 0$  (and complete the definition by linearity). Now  $P V_1 = V_1 = A$  on  $E_j$ .

The situation is thus that the eigenvalues of the operator  $P V_0 + A$  lie in the (closed) left half-plane for any "anti-symmetric"  $A$ . This is equivalent to the condition that  $Q(z) = \operatorname{Re} \langle V_0 z, z \rangle \leq 0$  whenever  $v(z) = \langle z, z \rangle \geq 0$  on  $E_j$  or  $v(z)Q(z) \leq 0$  for any sign of  $v(z)$ .

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